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Analysis and Synthesis of $1/f$ Processes via Shannon Wavelets

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Abstract— $1/f$ processes can be very useful in modeling processes with long-term correlation. We propose analysis and synthesis procedures to express these processes in terms of the Shannon wavelet. Unlike previous techniques, our analysis procedure generates uncorrelated decomposition coefficients for the $1/f$ process. This is done by taking onto account, and then removing, the residual correlation between the wavelet components. The analysis procedure is the major contribution of this work. The proposed synthesis algorithm, which is a byproduct of the proposed analysis algorithm, is competitive with other techniques.

Index Terms—Nonstationary processes, $1/f$ noise, spectral analysis, wavelets.

I. INTRODUCTION

Fractional Brownian motions, or $1/f$ processes, model well processes with long correlation time. However, in contrast with the standard models for stochastic processes, e.g., ARMA models, which fit well data with short-term correlation, the $1/f$ process is nonstationary [1]–[4]. Thus, the usual approach for analysis and synthesis in terms of sinusoidal waveforms (Fourier analysis), developed for stationary processes, does not fit the nonstationary $1/f$ process. For the comprehensive review on $1/f$ processes, see [4].

During the last two decades, various techniques for synthesis of $1/f$ processes appeared in the literature; see, for example, [3]–[9]. However, a complete and satisfactory solution to the analysis problem is still needed. Roughly speaking, the analysis problem is as follows. Given a random process with a known (or assumed) second-order properties, find the transformation that decorrelates the process. Under appropriate conditions, this transformation can be evaluated by solving an eigenvalue, eigenvector problem; this is the well-known Karhunen–Loève (K–L) transform. For stationary processes with long observation time, the K–L transform becomes the Fourier transform.

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In some other cases, the discrete cosine transform (DCT) approximates the K–L transform [10]. However, in many cases, including our case of $1/f$ processes, the calculation of the K–L transform is difficult and even impossible. Furthermore, $1/f$ processes are not stationary. Thus, if the K–L transform is calculated for short blocks, as often done in practice, it will be time dependent.

As shown in [1], [3], and [11], by using wavelets with an appropriate basic wavelet function, a nonstationary process can be decomposed into a number of stationary processes. These processes can be analyzed independently by tools developed for stationary processes and then combined again by applying the inverse Wavelet transform into the desired $1/f$ process.

In this work, we first show a simple way to calculate the correlation between the wavelet coefficients and apply the result to the $1/f$ processes. This is then used as the basis for a new technique for analysis and synthesis of the $1/f$ process, which takes into account the calculated residual correlation between the wavelet coefficients and use it to whiten the wavelet coefficients.

II. BACKGROUND

The usual definition of the power spectrum holds only for stationary processes. There exist, however, a number of definitions of a generalized power spectrum that are suitable for nonstationary processes, such as the time-averaged spectrum, which we use throughout the paper:

$$\bar{S}_x(\omega) = \int_{-\infty}^{\infty} E_t\{R_x(t, t + \tau)\} \exp^{-j\omega\tau} d\tau \quad (1)$$

where $R_x(t_1, t_2)$ is the correlation function of the process $x(t)$, $j = \sqrt{-1}$, and $E_t\{\cdot\}$ is the time averaging operation. The time-averaged spectrum of $1/f$ processes [4], [9] is given by

$$\bar{S}_x(\omega) = \frac{k}{|\omega|^\gamma} \quad (2)$$

where k is some positive constant, and γ is the process parameter, which raises difficulties at the origin and/or at infinity. This last problem, however, is not too severe since, in complete agreement with our measurement ability, we can assume that the process is viewed after passing through a bandpass filter with a lower cutoff frequency near the origin and an upper cutoff frequency as large as desired.

A powerful tool for dealing with the nonstationary nature of $1/f$ processes is the wavelet transform [12]. Our interest is restricted to the orthonormal dyadic wavelet expansions. The orthonormal dyadic wavelet transform of the signal $x(t)$ is expressed as

$$a_{mn} = \int_{-\infty}^{\infty} x(t) 2^{m/2} \psi(2^m t - n) dt \quad (3)$$

where $\psi(t)$ is a basic wavelet that satisfies some admissibility conditions [12]. Inversely, the process $x(t)$ can be represented as

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn} 2^{m/2} \psi(2^m t - n). \quad (4)$$

In our case, we are looking for a wavelet whose dilations do not overlap in frequency, i.e.,

$$\text{supp}\{\Psi(2^{-m}\omega)\} \cap \text{supp}\{\Psi(2^{-l}\omega)\} = \emptyset, \quad \forall l \neq m \quad (5)$$

where $\Psi(\omega)$ is a Fourier transform of $\psi(t)$. A wavelet function satisfying this property is the Shannon wavelet

$$\Psi(\omega) = \begin{cases} \frac{1}{2\pi} & \pi \leq |\omega| < 2\pi \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

The decomposition of a stationary signal produces, with any wavelet, a stationary output for which the power spectral density at a scale m will be

$$S_x(\omega; m) = 2^{-m} |\Psi(2^{-m}\omega)|^2 S_x(\omega) \quad (7)$$

where $S_x(\omega; m)$ denotes the power spectrum of the time process $x(t)$ at scale m . Note that if the wavelet $\psi(\omega)$ is chosen to be the Shannon wavelet, then (7) becomes

$$S_x(\omega; m) = \begin{cases} S_x(\omega) & 2^m\pi \leq |\omega| < 2^{m+1}\pi \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Under certain condition, which are stated in [1] and [11], the decomposition of a nonstationary process produces a stationary output, i.e., (7) and (8) can be applied to the time-averaged spectrum of the nonstationary process. It is interesting to note that this property leads to a natural definition of $1/f$ processes, as shown in [4].

III. MAIN RESULTS

We begin by presenting an expression for the correlation function of the wavelet decomposition coefficients and apply this expression for $1/f$ processes. This result appears in more detail in [4], [6], [13], and [14], but since our procedure is based on this expression, it is repeated here. Then, we present our analysis/synthesis algorithms for $1/f$ processes.

A. Correlation Between the Wavelet Decomposition Components

Under certain conditions, the decomposition of a nonstationary process produces a stationary output. Specifically, as shown in [11], the condition for stationarity of the wavelet decomposition output is $r \geq D - 1$, where r is the number of vanishing moments of the basic wavelet, and D is the order of the input process with stationary increments as defined in [11]. It is easy to check that the order of the $1/f$ process is $D = 1$. Thus, the decomposition of the $1/f$ process $x(t)$ with any wavelet will produce stationary output, for which the power spectral density at a scale m , which is denoted $S_x(\omega; m)$, will be

$$S_x(\omega; m) = 2^{-m} |\Psi(2^{-m}\omega)|^2 \bar{S}_x(\omega). \quad (9)$$

On the other hand, since the number of vanishing moments for an ideal Shannon wavelet tends to infinity, the Shannon wavelet decomposition is sufficient to decompose any $1/f$ process into a number of stationary processes.

Let $R(m, k, n, l) = E\{a_{mn} a_{kl}^*\}$. Using the assumption that the basic wavelet is strictly localized and that various wavelet decomposition components do not overlap in frequency [see (6)], the power spectrum on the interval $[-2^{m+1}\pi, -2^m\pi] \cup [2^m\pi, 2^{m+1}\pi]$ is

$$S_x(\omega; m) = \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} R(m, n, l) \exp\{-j\omega 2^{-m}(n-l)\}. \quad (10)$$

As mentioned previously, the Shannon wavelet decomposition of the continuous-time $1/f$ process produces stationary outputs. Thus, (10) becomes

$$S_x(\omega; m) = \sum_{n=-\infty}^{\infty} R(m, |n|) \cos(\omega 2^{-m} n). \quad (11)$$

On the given interval, the set $\{\cos(2^{-m}\omega n)\}_{n \in \mathbb{Z}}$ is an orthogonal family of functions; therefore, the normalized autocorrelation function of the components for any finite numbers m, n can be calculated by

$$\begin{aligned} \rho(m, |n|) &= \frac{R(m, |n|)}{R(m, 0)} \\ &= \begin{cases} 2 \frac{\int_{\pi}^{2\pi} \bar{S}_x(2^m \omega) \cos(n\omega) d\omega}{\int_{\pi}^{2\pi} \bar{S}_x(2^m \omega) d\omega} & n \neq 0 \\ 1 & n = 0. \end{cases} \end{aligned} \quad (12)$$

It should be pointed out that (12) holds only for the Shannon wavelet (rectangular in frequency). However, these equations can give a reasonable estimate of the correlation between the wavelet coefficients for a class of wavelets whose shape is close to the shape of the Shannon wavelet.

B. New Analysis and Synthesis Procedures

In this subsection, we propose a solution to the analysis problem using two main observations. The first is the fact that the wavelet decomposition of a nonstationary signal produces a number of stationary signals. The second observation is that the same autocorrelation function describes the behavior of the wavelet coefficients at all scales. Substituting the expression for time-averaged spectrum of the $1/f$ process and the Shannon wavelet into (12), we get

$$\begin{aligned} \rho(m, n) &= \frac{R(m, n)}{R(m, 0)} \\ &= \begin{cases} 2 \frac{\int_{\pi}^{2\pi} \omega^{-\gamma} \cos(n\omega) d\omega}{\int_{\pi}^{2\pi} \omega^{-\gamma} d\omega} & n \neq 0 \\ 1 & n = 0. \end{cases} \end{aligned} \quad (13)$$

As expected, for $1/f$ processes, this ratio does not depend on m , i.e., the correlation of components is the same for all levels of decomposition. The integral in the numerator is difficult to evaluate analytically, but it can be evaluated numerically. The result for two values of γ is given in the second row of Tables I and II.

Once the correlation function of the process is known, the analysis problem, whose purpose is to generate uncorrelated components, can be solved in many ways, for instance, by using a K-L transformation that fits this correlation function. Another option is to whiten the stationary signal by a causal whitening filter of an appropriate order, which is found from the correlation function by solving the Yule-Walker equations for the autoregressive (AR) order n_0 case

$$\begin{bmatrix} 1 & \rho(1) & \cdots & \rho(n_0) \\ \rho(1) & 1 & \vdots & \rho(n_0 - 1) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(n_0) & \rho(n_0 - 1) & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \\ \vdots \\ a(n_0) \end{bmatrix} = \begin{pmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (14)$$

where σ^2 is the variance of the whitened process, and n_0 is chosen such that $\rho(n_0) \ll 1$. We have chosen this option. It should be emphasized that since the correlation function of the wavelet components is independent of the scale m in the particular case of

TABLE I
ESTIMATED WAVELET COMPONENTS CORRELATION COEFFICIENTS FOR $\gamma = 1$

Algorithm	tap index					
	0	1	2	3	4	5
Shannon wavelet with whitening filter	1	-.0010	.0008	-.0084	-.0127	-.0060
Shannon wavelet without whitening filter	1	-.1399	.0257	-.0263	-.0040	-.0125
Daubeshies 48 tap filter with whitening filter	1	.0213	.0026	-.0232	-.0027	-.0074
Daubeshies 48 tap filter without whitening filter	1	-.1509	.0343	-.0253	.0027	-.0091

TABLE II
ESTIMATED WAVELET COMPONENTS CORRELATION COEFFICIENTS FOR $\gamma = 2$

Algorithm	tap index					
	0	1	2	3	4	5
Shannon wavelet with whitening filter	1	.0083	-.0009	-.0075	-.0004	.0001
Shannon wavelet without whitening filter	1	-.2668	.0685	-.0519	.0209	-.0145
Daubeshies 48 tap filter with whitening filter	1	.0694	.0149	-.0419	.0096	-.0141
Daubeshies 48 tap filter without whitening filter	1	-.2765	.0682	-.0498	.0300	-.0194

$1/f$ processes, it is sufficient to solve (13) and (14) only once, and the solution gives a desired whitening filter for all bands.

Now, let us discuss the properties of the whitening filter. Since the spectrum $S_x(\omega; m)$ corresponding to the band m of the decomposed $1/f$ process is strictly positive, the calculated correlation is a strictly positive definite function, and the solution of the Yule-Walker $n_0 + 1$ linear equations gives a minimum phase or, in other words, a causal and stable filter. Following this conclusion, the synthesis filter exists and is also causal and stable. The synthesis filter is given by $B(\omega) = 1/A(\omega)$, where $A(\omega)$ is the whitening filter.

In the presented solution, the derived synthesis filter is not a finite impulse response (FIR) filter. An alternative solution in which both the analysis and the synthesis filters are FIR filters follows. Find the whitening filter by solving (14), and find the synthesis filter by solving the Yule-Walker equations for the moving-average (MA) order n_0 case

$$\begin{bmatrix} 1 \\ \rho(1) \\ \vdots \\ \rho(n_0) \end{bmatrix} = \begin{bmatrix} b(0) & b(1) & \cdots & b(n_0-1) & b(n_0) \\ b(1) & \cdots & b(n_0-1) & b(n_0) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b(n_0) & 0 & \cdots & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} b(0)^* \\ b(1)^* \\ \vdots \\ b(n_0)^* \end{bmatrix}. \quad (15)$$

The solution of these nonlinear equations is more complicated and, in general, not unique. The problem arises because in the MA case, we assume that the correlation function is strictly zero outside the given interval, and this assumption can give a correlation function that is not positive definite. Thus, when the parameter n_0 is chosen, it must be verified that the resulting correlation function with zero extension is a positive definite function. If it is, then the solution will give a stable synthesis filter.

The whitening filter $A(\omega)$ can be combined with the decomposition filter using Noble identities and/or polyphase decomposition

[15]–[17], and we get a modified filter for the decomposition. The same is true for the synthesis filter. Three possible implementations of the wavelet transform followed by a whitening filter are shown in Fig. 1. The third (bottom) scheme relates to the polyphase decomposition of the decomposition filter, where the filter is represented as

$$\begin{aligned} G(\omega) &= \sum_{k=0}^{N-1} g(k) e^{-j\omega k} \\ &= \sum_{k=0}^{(N-1)/2} g(2k) e^{-j\omega 2k} \\ &\quad + e^{-j\omega} \sum_{k=0}^{(N-1)/2} g(2k+1) e^{-j\omega 2k} \\ &= E_0(2\omega) + e^{-j\omega} E_1(2\omega). \end{aligned} \quad (16)$$

The two bottom schemes in Fig. 1 can be regarded as one stage of a subband decomposition/reconstruction with a lowpass filter $H(\omega)$ and a highpass filter $C(\omega) = G(\omega)A(2\omega)$ [$\tilde{H}(\omega)$ and $\tilde{C}(\omega) = \tilde{G}(\omega)B(2\omega)$ in the reconstruction]. This decomposition may be equivalent to some other wavelet decomposition, with another, possibly nonorthogonal, wavelet base. In [8], it is shown how to find such a wavelet base that produces the desirable $1/f$ process. There is probably a relation between the decomposition filters proposed in our work and the wavelet bases found in [8]; however, this subject is still to be investigated.

Usually, γ is an unknown parameter. Several algorithms can be used to estimate it: for example, the EM algorithm described in [18]. Most of these algorithms include the wavelet transform as a preprocessing stage. We present the analysis procedures for both cases, i.e., for known and unknown γ :

- Known γ —

- a) Calculate the normalized correlation function of the wavelet coefficients (13) for some given m . This func-

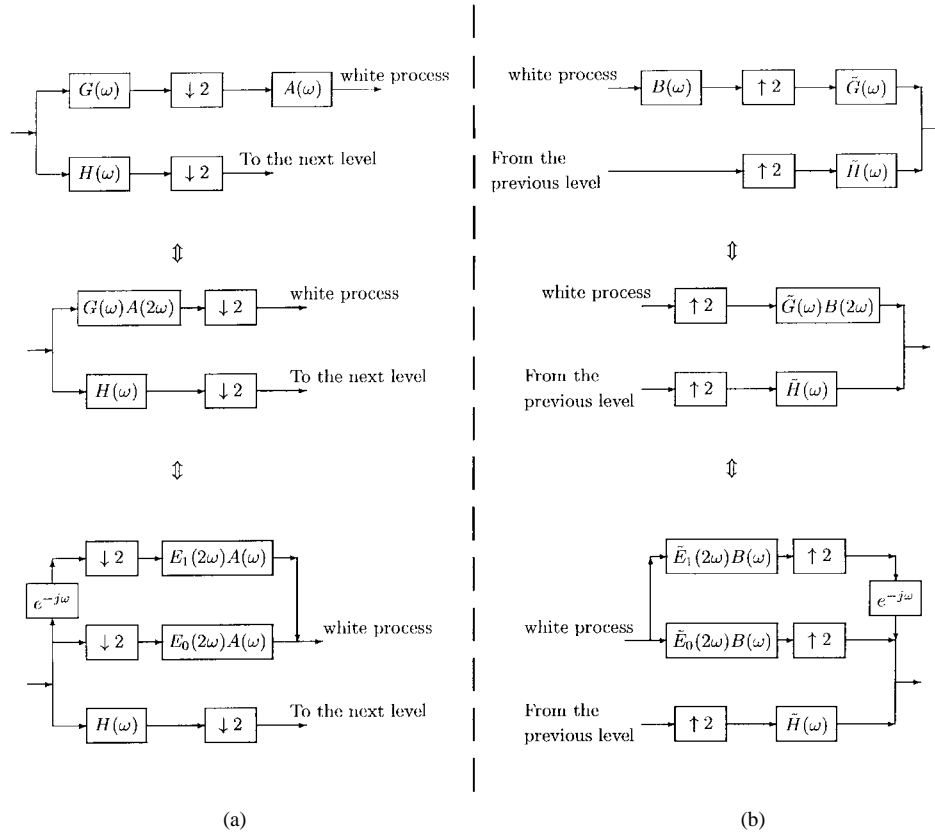


Fig. 1. Possible implementations of one stage wavelet decomposition/reconstruction.

tion can actually be calculated up to a value at lag n_0 , where $\rho(n_0; m) \ll 1$.

- b) Solve the Yule-Walker equations for the correlation function in Step 1, and get the whitening filter.
- c) Use the result of Step 2 to modify the highpass filter $G(\omega)$ of the wavelet decomposition.
- d) Perform the new wavelet decomposition.

• Unknown γ —

- a) Perform the wavelet decomposition.
- b) Estimate γ with the algorithm described in [18].
- c) Calculate the normalized correlation function of the wavelet coefficients (13) for some given m .
- d) Solve the Yule-Walker equations for the correlation function in Step 3, and get the whitening filter.
- e) Filter the output of the wavelet decomposition with the whitening filter.

The synthesis procedure is simpler. The correlation function for a given value of γ and then the whitening filter are calculated. The inverse wavelet decomposition filter and the inverse of the whitening filter are combined together. Now, passing white noise into this filter will produce the desired $1/f$ process at the output.

IV. SIMULATION RESULTS

In this section, some simulation results for the synthesis and analysis algorithms are presented and compared with the algorithm presented in [9]. Since the Shannon wavelet corresponds to an ideal, unrealizable filter, we may claim that this work is of theoretical interest only. However, the use of the Shannon wavelet simplifies the problem and gives valuable insights. In the computer implementation, the Shannon wavelet can be approximated by a long FIR filter or

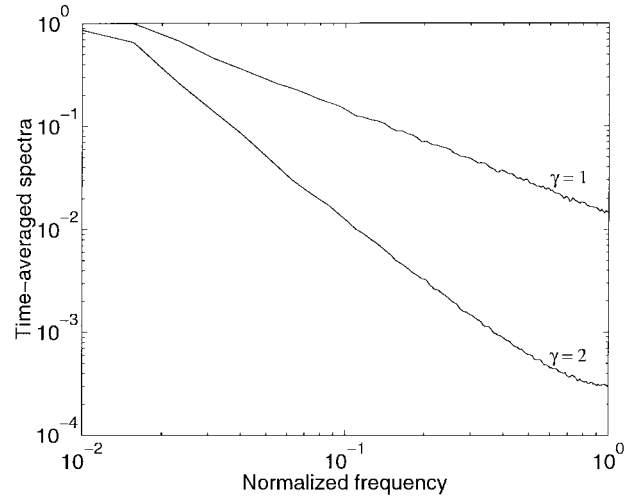


Fig. 2. Time-average spectra of analyzed signals.

by a rectangular window in the frequency domain. Furthermore, Daubechies wavelets [12] approximate the Shannon wavelet as their *smoothness index* N tends to infinity. Since the option to analyze $1/f$ data with a relatively short FIR filter is an important issue, we also test Daubechies wavelets with 48 taps (corresponds to $N = 24$) and obtain that they give a satisfactory results in terms of the wavelet components correlation. Note that in this case, (13) does not hold, and an alternative equation was derived from (12) by replacing $\bar{S}_x(2^m \omega)$ with $|\Psi(\omega)|\bar{S}_x(2^m \omega)$ and neglecting the small overlap between adjacent bands. The number of data points in the analyzed and synthesized signals in our simulation was 100 000.

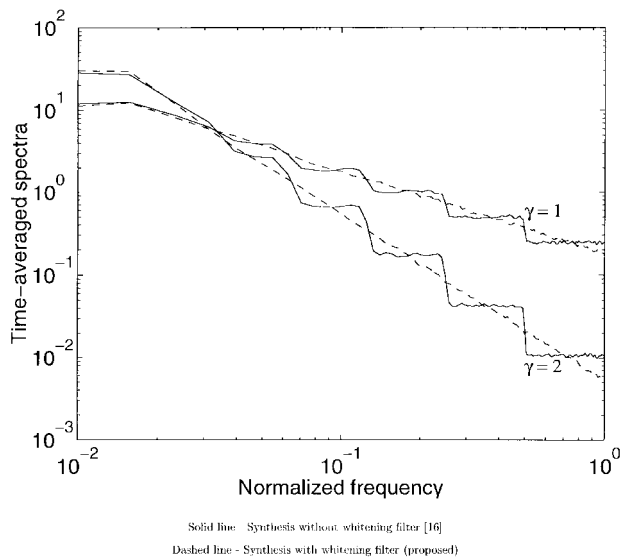


Fig. 3. Time-average spectra of a synthetic signal using ideal filter approximation.

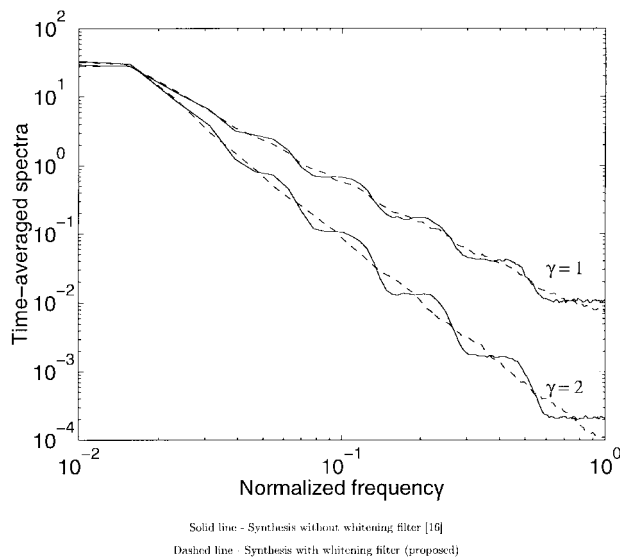


Fig. 4. Time-average spectra of a synthetic signal using Daubechies 48 tap filter.

A. Analysis

The first analyzed signal was synthesized by the algorithm presented in [7], i.e., the $1/f$ process was created by simulating an infinite RC line (corresponds to $\gamma = 1$). The second analyzed signal was synthesized by simulating Brownian motion (corresponds to $\gamma = 2$). The time-average spectra of the input processes are presented in Fig. 2.

The performance of the algorithms was tested by estimating the decomposition components correlation function and comparing with the analysis procedure, which does not use the whitening filter. Tables I and II show that when the whitening filter is used, the analysis filter bank gives almost uncorrelated output. These tables demonstrate the improved quality of the proposed analysis algorithm. Note that the tables are given only for bands with index 0, but all other bands have a similar behavior. In the analysis procedure with unknown γ , the estimate of γ was done by the algorithm proposed

in [18]. The difference between the analysis results with known γ was negligible.

B. Synthesis

The algorithm proposed in this work was compared with the algorithm proposed in [9]. Both algorithms were implemented with two different wavelet bases—the Shannon wavelet and Daubechies wavelets [12]. The time-average spectra of the synthesized processes are shown in Figs. 3 and 4. From these figures, we can see that the time-average spectrum of the signal produced by the procedure proposed in this paper has the most accurate $1/f$ behavior.

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